Flux Saturation Length of Sediment Transport

Thomas Pähtz, Jasper F. Kok, Eric J. R. Parteli, and Hans J. Herrmann

1Department of Ocean Science and Engineering, Zhejiang University, 310058 Hangzhou, China
2State Key Laboratory of Satellite Ocean Environment Dynamics, Second Institute of Oceanography, 310012 Hangzhou, China
3Department of Atmospheric and Oceanic Sciences, University of California, Los Angeles, California 90095, USA
4Institute for Multiscale Simulation, Universität Erlangen-Nürnberg, Nögelsbachstrasse 49h, 91052 Erlangen, Germany
5Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil
6School of Computer Science, Zhejiang Normal University, 321004 Jinhua, Zhejiang, China
7Computational Physics, IJF, ETH Zürich, Schafmattstrasse 6, 8093 Zürich, Switzerland

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Sediment transport along the surface drives geophysical phenomena as diverse as wind erosion and dune formation. The main length scale controlling the dynamics of sediment erosion and deposition is the saturation length \( L_s \), which characterizes the flux response to a change in transport conditions. Here we derive, for the first time, an expression predicting \( L_s \) as a function of the average sediment velocity under different physical environments. Our expression accounts for both the characteristics of sediment entrainment and the saturation of particle and fluid velocities, and has only two physical parameters which can be estimated directly from independent experiments. We show that our expression is consistent with measurements of \( L_s \) in both aeolian and subaqueous transport regimes over at least 5 orders of magnitude in the ratio of fluid and particle density, including on Mars.

\[ \Gamma(Q) = dQ/dx \cong (Q_s - Q)/L_s, \]

where \( M \) is the mass of sediment in flow per unit soil area, and \( V \) is the average particle velocity. Since the fluid loses momentum to accelerate the particles, \( Q \) is limited by a steady-state value, the saturated flux \( Q_s \). This flux is largely set by the fluid density \( \rho_f \) and the fluid shear velocity \( u_s \) [1–4,6], which is proportional to the mean flow velocity gradient in turbulent boundary layer flow [6]. In typical situations, such as on the streamward side of dunes, the deviation of \( Q \) from \( Q_s \) is small, that is, \( |1 - Q/Q_s| \ll 1 \) [10,11,16]. The rate \( \Gamma(Q) \) of the relaxation of \( Q \) towards \( Q_s \) in the downstream direction \( x \) can thus be approximately written as [7,10,16],

\[ \Gamma(Q) = dQ/dx \cong (Q_s - Q)/L_s, \]

where \( \Gamma \) is Taylor expanded to first order around \( Q = Q_s \) [\( \Gamma(Q_s) = 0 \)], and the negative inverse Taylor coefficient gives the saturation length, \( L_s \). Flux saturation is controlled by the downstream evolutions of \( M \) and \( V \) towards their respective steady-state values, \( M_s \) and \( V_s \). Changes in \( M \) with \( x \) are controlled by particle entrainment from the sediment bed into the transport layer. In the aeolian regime (dilute fluid such as air), entrainment occurs predominantly through particle impacts [6], whereas in the subaqueous regime (dense fluid such as water) entrainment occurs mainly through fluid lifting [2,3]. On the other hand, the evolution of \( V \) towards \( V_s \) is mainly controlled by the acceleration of the particles due to fluid drag, and their deceleration due to grain-bed collisions [10,12]. We note that the evolution of \( V \) is affected by changes in \( M \) and vice versa. For instance, an increase of \( M \) leads to a decrease in \( V \) in the absence of horizontal forces due to conservation of horizontal momentum. For simplicity, previous studies...
neglected either the saturation of $V$ [10,17] or the relaxation of $M$, as well as changes in $V$ due to grain-bed collisions [7,12]. Moreover, all previous studies did not account for the relaxation of the fluid velocity ($U$) towards its steady-state value ($U_s$) within the transport layer. This relaxation is driven by changes in the transport-flow feedback resulting from the relaxations of $M$ and $V$. For instance, increasing $V$ reduces the relative velocity $V_r = U - V$ and thus the fluid drag. In turn, as $V_r$ decreases, the amount of momentum transferred from the fluid to the transport layer also decreases, which results in an increase in $U$, whereas an increase in $U$ again increases $V_r$.

In this Letter, we derive a theoretical expression for $L_s$ which encodes all aforementioned relaxation mechanisms. Indeed, since previously proposed relations for $L_s$ neglect some of the interactions that determine $L_s$ [7,10–12], it is uncertain how to adapt these equations to compute $L_s$ in extraterrestrial environments, such as Mars [5,6,13]. Our theoretical expression overcomes this problem, since it is valid for arbitrary physical environments for which turbulent fluctuations of the fluid velocity, and thus transport as suspended load [6], can be neglected. For aeolian transport under terrestrial conditions, this regime corresponds to $u_s \approx 4u_f$, where $u_s$ is the threshold $u_s$ for sustained transport [2,3,6].

We start from the momentum conservation equation for steady ($\partial / \partial t = 0$) dilute granular flows [18],

$$\frac{\partial \langle \rho v_x^2 \rangle}{\partial x} + \frac{\partial \langle \rho v_y v_z \rangle}{\partial z} = \langle f_x \rangle,$$  

(2)

where $\langle \cdot \rangle$ denotes the ensemble average, $\rho$ the mass density, $v$ the particle velocity, and $f$ the external body force per unit volume applied on a sediment particle. Here, $f$ incorporates the main external forces acting on the transported particles: drag, gravity, buoyancy, and added mass. The added mass force arises because the speed of the fluid layer immediately surrounding the particle is closely coupled to that of the particle, thereby enhancing the particle’s inertia by a factor $1 + 0.5s^{-1}$, where $s = \rho_j / \rho_f$ is the grain-fluid density ratio [2]. Although this added mass effect is negligible in aeolian transport ($0.5s^{-1} \ll 1$), it affects the motion of particles in the subaqueous regime [2]. Integration of Eq. (2) over the entire transport layer depth ($\int_0^\infty dz$) yields,

$$\frac{d\langle c_v M V^2 \rangle}{dx} = \int_0^\infty \langle f_x \rangle dz + \langle \rho v_y v_z \rangle(0).$$  

(3)

where $M = \int_0^\infty \rho dz$, $V = \int_0^\infty \rho v_y dz / M$, and $c_v = \int_0^\infty \rho v_y^2 dz / (MV^2)$. In Eq. (3), the quantity $\langle \rho v_y v_z \rangle(0)$ gives the difference between the average horizontal momentum of particles impacting onto $[-(\rho_l v_y v_z)](0)$ and leaving $[(\rho_l v_y v_z)](0)$ the sediment bed per unit time and soil area. This momentum change is consequence of the collisions between particles within the sediment bed ($z \leq 0$). Thus, $\langle \rho v_y v_z \rangle(0)$ is an effective friction force which the soil applies on the transport layer per unit soil area. It is proportional to the normal component of the force which the transport layer exerts onto the sediment bed [3,10,19], $\langle \rho(v_y,v_z) \rangle(0) = -\mu g M (s-1)/(s+0.5)$, where $\mu$ is the associated Coulomb friction coefficient, and $g$ the gravitational constant. In order to obtain the momentum conservation equation of the particles within the transport layer from Eq. (3), we first note that $\int_0^\infty \langle f_x \rangle dz = (3M/4s)dC_d(V_r) V_r^2$ [19], where $d$ is the mean grain diameter, while $C_d(V_r)$ is the drag coefficient associated with the fluid drag on transported particles, which is intermediate to fully viscous drag ($C_d \propto V/[V_D, v]$), with $V$ standing for the kinematic viscosity) and fully turbulent drag (constant $C_d$). By further noting that the change of $c_v$ with $x$ is negligible (see Supplemental Material [20]), we obtain,

$$c_v \frac{d(MV^2)}{dx} = \frac{3M}{4(s+0.5)d} C_d(V_r)V_r^2 - \frac{s-1}{s+0.5} \mu g M.$$  

(4)

Next, we solve Eq. (4) for $dV/dx$ thus obtaining an equation of the form $(dV/dx) = \Omega (V)$, and we expand $\Omega (V)$ around saturation, that is, $\Omega (V) = \Omega (V_s) + (V - V_s)d\Omega / dV |_{V_s}$. By noting that $\Gamma (V) = (dQ/dx)(V) = (M(V) + V(dM(V)/dV))\Omega (V)$ and $\Omega (V_s) = 0$, we obtain $L_s = -(d\Gamma / dQ)^{-1}_0 \Gamma = -(d\Omega / dV)^{-1}_0 \Omega |_{V=V_s}$, which leads to,

$$L_s = (s+0.5)c_v(2 + c_M)V_s F K [\mu (s-1) g]^{-1}.$$  

(5)

where $c_M = (V_s/M_s)(dM/dV)(V_s)$, and $K = (1 - (d\Omega/dV(V_s)))^{-1}$, while $V_s$ (the steady-state value of $V_s$) and $F$ are given by,

$$V_s = \sqrt{\frac{8\mu (s-1) g d / 9 + (8\nu / d)^2 - 8\nu / d}} \text{ and,}$$  

$$F = [V_s + 16\nu / d] [2V_s + 16\nu / d]^{-1}.$$  

(6)

respectively. Equations (6) and (7) result from using $C_d(V) = (24\nu / V_R d) + 1.5$ (valid for natural sediment [21]). We find that using other reported drag laws only marginally affects the value of $L_s$. Furthermore, we note that in the subaqueous regime $c_M = 0$, since in this regime $M$ changes within a time scale which is more than 1 order of magnitude larger than the time scale over which $Q$ changes [22]. This difference in time scales implies $VdM \ll dQ$ and thus $VdM \ll MdV$ in the subaqueous regime. In contrast, in the aeolian regime, $c_M = 1$ as the total mass of ejected grains upon grain-bed collisions is approximately proportional to the speed of impacting grains [23], which yields $M/M_s = V/V_s$.

In Eq. (5), the quantity $K$ encodes the effect of the relaxation of the transport-flow feedback, neglected in previous works [7,10,17]. In the subaqueous regime, this transport-flow feedback has a negligible influence on the fluid speed [22] (and thus on its relaxation). In this regime, $(dU/dV)(V_s) = 0$, which yields $K = 1$ and thus,
\[ L_{s}^{\text{subaq}} = [2s + 1]c_{s}V_{s}V_{r}F[\mu(s - 1)g]^{-1}. \]  

In contrast, in the aeolian regime, \( U \) scales with the shear velocity at the bed \( (u_{b}) \) [19,22], and thus \( (dU/dV)(V_{s}) = (U_{s}/u_{b})du_{b}/dV(V_{s}) \), where \( u_{b} \) is the steady-state value of \( u_{s} \). Using the mixing length approximation of inner turbulent boundary layer equations [24], \( u_{b} \) can be expressed as \[ u_{b} = u_{s}[1 - 3MC_{d}(V_{s})V_{r}^{2}/(4(s + 0.5)dp_{f}u_{s}^{1.5})]^{1/2} \] [22]. By using this expression to compute \( du_{b}/dV \) and noting that \( u_{b} \approx u_{t} \) [6], we obtain the following expression for \( K \),

\[ K = \frac{1 + F^{-1}[V_{s} + V_{r}]/(2V_{r})][(u_{s}/u_{t})^{2} - 1]}{1 + [(V_{s} + V_{r})/(2V_{r})][(u_{s}/u_{t})^{2} - 1]}. \]

Using Eq. (9) to compute \( K, L_{s} \) in the aeolian regime of transport \([s + 0.5]/(s - 1) \approx 1\) is then given by

\[ L_{s}^{\text{aeolian}} = 3c_{w}V_{s}V_{r}F[\mu g]^{-1}. \]  

We show in Section IV of the Supplemental Material [20] that Eq. (10) can be approximated by the simpler form of \[ L_{s}^{\text{aeolian}} = 3c_{w}V_{s}^{2}F[\mu g]^{-1} \] in the limit of large \( u_{s}/u_{t} \).

Therefore, from our general expression for \( L_{s} \) [Eq. (5)] we obtain two expressions—Eqs. (8) and (10)—which can be used to predict \( L_{s} \) in the subaqueous and aeolian transport regimes, respectively. Both use only two parameters, namely \( \mu \) and \( c_{w} \), which are estimated from independent measurements. Specifically, \( \mu \) is estimated from measurements of \( M_{f} \) and \( Q_{s} \) for different values of \( u_{s} \) in air and under water, while \( c_{w} \) is estimated from measurements of the particle velocity distribution [20,25,26]. From these experimental data, we obtain \( \mu = 1.0 \) (0.5) and \( c_{w} = 1.3 \) (1.7) for the aeolian (subaqueous) regime.

Both Eqs. (8) and (10) are consistent with the behavior of \( L_{s} \) with \( u_{s} \) observed in experiments. Indeed, \( L_{s} \) mainly depends on \( u_{s} \) via the average particle velocity, \( V_{s} \). For subaqueous transport, in which \( V_{s} \) is a linear function of \( u_{s} \), \( L_{s} \) varies linearly with \( V_{s} \) and thus with \( u_{s} \), which is consistent with experiments [8]. In contrast, \( V_{s} \) depends only weakly on \( u_{s} \) for aeolian transport [6,22]. Consequently, \( L_{s} \) is only weakly dependent on \( u_{s} \) in this regime, which is also consistent with experiments [7]. In fact, when neglecting this weak dependence on \( u_{s} \), Eq. (10) reduces to \( L_{s} \sim sd \) [7,12] in the limit of large particle Reynolds numbers \( \sqrt{sgd^{3}/\nu} \) for which \( V_{s} \sim \sqrt{sgd} \) [22]. Moreover, we estimate the average particle velocity \( V_{s} \) as a function of \( u_{s}/u_{t} \) using well-established theoretical expressions which were validated against experiments of sediment transport in the aeolian or in the subaqueous regime. Specifically, we use the model of Ref. [19] for obtaining \( V_{s}(u_{s}/u_{t}) \) in the aeolian regime and the model of Ref. [25] for the subaqueous regime [20].

The squares in Fig. 1 denote wind tunnel measurements of \( L_{s} \) for different values of \( u_{s} \). These data were obtained by fitting Eq. (1) to the downstream evolution of the sediment flux, \( Q(x) \), close to equilibrium [7]. Further estimates of \( L_{s} \) for aeolian transport under terrestrial conditions have been obtained from the wavelength (\( \lambda \)) of elementary dunes on top of large barchans [7,20]. These estimates correspond to the circles in Fig. 1, whereas the squares in this figure denote \( L_{s} \) versus \( u_{s}/u_{t} \) for aeolian transport under terrestrial conditions. Brown squares denote estimates of \( L_{s} \) from wind-tunnel measurements (\( d = 120 \mu m \)), while the error bars are due to uncertainties in the measurements of the sediment flux [7]. Green circles denote \( L_{s} \) obtained from the wavelength of elementary dunes on top of large barchans (\( d = 185 \mu m \)), whereas the error bars contain uncertainties in the dune size [7] (potential systematic uncertainties [20] are not included). The colored lines represent predicted values of \( L_{s} \) using Eq. (5) for the corresponding experimental conditions (\( \rho_{p} = 2650 \text{ kg/m}^{3} \), \( \rho_{f} = 1.174 \text{ kg/m}^{3} \) and \( \nu = 1.59 \times 10^{-5} \text{ m}^{2}/\text{s} \)). The dotted horizontal line indicates the prediction of \( L_{s} \) using \( L_{s} = 2sd \) [7,12]. The upper legend displays the corresponding values of the coefficient of determination, \( R^{2} = 1 - (\sum_{i}(L_{s}^{\text{measured}} - L_{s}^{\text{predicted}})^{2}/\sum_{i}(L_{s}^{\text{measured}} - \bar{L}_{s}^{\text{mean}})^{2}) \), which is a measure of a theory’s ability to capture variation in data, with \( R^{2} = 1 \) corresponding to a perfect fit.

FIG. 1 (color online). Dimensionless saturation length, \( L_{s}/(sd) \), versus \( u_{s}/u_{t} \) for aeolian transport under terrestrial conditions. Brown squares denote estimates of \( L_{s} \) from wind-tunnel measurements (\( d = 120 \mu m \)), while the error bars are due to uncertainties in the measurements of the sediment flux [7]. Green circles denote \( L_{s} \) obtained from the wavelength of elementary dunes on top of large barchans (\( d = 185 \mu m \)), whereas the error bars contain uncertainties in the dune size [7] (potential systematic uncertainties [20] are not included). The colored lines represent predicted values of \( L_{s} \) using Eq. (5) for the corresponding experimental conditions (\( \rho_{p} = 2650 \text{ kg/m}^{3} \), \( \rho_{f} = 1.174 \text{ kg/m}^{3} \) and \( \nu = 1.59 \times 10^{-5} \text{ m}^{2}/\text{s} \)). The dotted horizontal line indicates the prediction of \( L_{s} \) using \( L_{s} = 2sd \) [7,12]. The upper legend displays the corresponding values of the coefficient of determination, \( R^{2} = 1 - (\sum_{i}(L_{s}^{\text{measured}} - L_{s}^{\text{predicted}})^{2}/\sum_{i}(L_{s}^{\text{measured}} - \bar{L}_{s}^{\text{mean}})^{2}) \), which is a measure of a theory’s ability to capture variation in data, with \( R^{2} = 1 \) corresponding to a perfect fit.
FIG. 2 (color online). $L_s/(sd)$ versus $u_s/u_t$ for subaqueous transport. Symbols denote estimates of $L_s$ from the wavelength of elementary dunes [12] and from the minimal cross-stream width of subaqueous barchans, $W = 12L_s$ [8]. The colored lines denote predicted values of $L_s$ using Eq. (5) for subaqueous transport of sand ($\rho_s = 2650 \text{ kg/m}^3$, $\rho_f = 10^3 \text{ kg/m}^3$ and $v = 10^{-6} \text{ m}^2/\text{s}$), with grain sizes roughly matching those used in the experiments. The dotted horizontal line indicates the prediction of $L_s$ using the scaling $L_s = 2sd$ [7,12]. The values of $R^2$ (coefficient of determination) for both expressions are also shown.

An excellent laboratory for further testing our model is the surface of Mars, where the ratio of grain to fluid density ($s$) is about 2 orders of magnitude larger than on Earth. We estimate the Martian $L_s$ from reported values of the minimal crosswind width $W$ of barchans at the Arkhangelsky crater in the southern highlands and at a dune field near the north pole [13,20]. However, using Eq. (10) to predict $L_s$ on Mars is difficult because both the grain size $d$ and the typical shear velocity $u_{s,typ}$ for which the dunes were formed are poorly known. Indeed, we need to know both quantities to calculate $V_s$ [19]. We thus predict the Martian $L_s$ using a range of plausible values of $d$ and $u_{s,typ}$. Specifically, we assume $d$ to lie in the broad range of 100–600 $\mu$m based on recent studies [5]. Estimating $u_{s,typ}$ on Mars is also difficult, both because of the scarcity of wind speed measurements [28], and because the threshold $u_t$ required to initiate transport ($u_{f,t}$) likely exceeds $u_t$ by up to a factor of $\sim$10 [6,19,29]. We therefore calculate $L_s$ for two separate estimates of $u_{s,typ}$: the first using $u_{s,typ} = u_{f,t}$, consistent with previous studies [13,30], and the second calculating $u_{s,typ}$ based on the wind speed probability distribution measured at the Viking 2 landing site [20], which results in an estimate of $u_{s,typ}$ closer to $u_t$. Figure 3 shows that the values of $L_s$ predicted with either of these estimates are consistent with those estimated from the minimal barchan width. This good agreement suggests that the previously noted overestimation of the minimal size of Martian dunes [31] is largely resolved by accounting for the low Martian value of $u_t/u_{f,t}$ [19] and the proportionally lower value of the particle speed $V_s$, as hypothesized in Ref. [29]. Indeed, the scaling $L_s = 2sd$ (inset of Fig. 3) requires $d = 29 \mu$m and $d = 40 \mu$m to be consistent with $L_s$ for the north pole and Arkhangelsky dune fields, respectively. However, such particles are most likely transported as a suspended load on Mars [30], as they are on Earth [4].

Finally, Fig. 3 also compares Eq. (8) to measurements of $L_s$ for Venusian transport, which have been estimated from the wavelength of elementary dunes produced in a wind-tunnel mimicking the Venusian atmosphere [32].

In conclusion, Eq. (5) is the first expression capable of quantitatively reproducing measurements of the saturation length $L_s$ under different flow conditions in both air and under water, and is in agreement with measurements over at least 5 orders of magnitude of variation in the sediment to fluid density ratio. The future application of this expression thus has the potential to provide important contributions to...
calculate sediment transport, the response of saltation-driven wind erosion and dust aerosol emission to turbulent wind fluctuations, and the dynamics of sediment-composed landscapes under water, on Earth’s surface and on other planetary bodies. The code to calculate $L_s$ with our model is available from the first author.

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*Corresponding author.
0012136@zju.edu.cn

[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.111.218002 for a description of how we estimated the model parameters $c_v$ and $\mu$, the average sediment velocity $V_s$, and the saturation length of sediment transport from the size of dunes under water and on planetary bodies.